Transmission Line Model

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Abstract

This document describes an overhead transmission line model that is useful for the analysis of large scale electric power systems. It establishes practical techniques for computing the series impedance and shunt admittance of arbitrary conductor configurations. Consideration is given to computational aspects of computing transmission line impedance parameters.

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1 Introduction

Transmission and distribution lines consist of an arbitrary spatial arrangement of one or more conductors. Information about these conductors is transformed into parameters required for power system analysis as follows:

- The fundamental data consists of a description of each conductor and how the conductors are arranged on their support structures.
- Conductor and spacing information is converted into an impedance matrix representing the self and mutual impedances of the complete configuration.
- The impedance matrix is reduced to eliminate elements that are not required by the analysis.
- The reduced impedance matrix is converted to symmetrical components when sequence impedances are required.

If sequence impedances are the only available information, they can be transformed into a reduced impedance matrix.

The remainder of this document examines the first two stages of this modeling process in detail. That is, we examine techniques for transforming conductor parameters and configuration data into impedance and capacitance matrices. The analysis is limited to overhead transmission lines.

2 Series Impedance

The series impedance of an overhead transmission line is primarily a function of frequency, conductor resistance, conductor geometry, line geometry, and earth conductivity. In the fundamental work on the subject, Carson (1923) developed equations for the self impedance of a conductor with earth return and the mutual impedance of two conductors with common earth return. These equations have been discussed and elaborated upon many times over the years. Wagner and Evans (1933), Clarke (1943), and Anderson (1987) provide excellent complementary discussions of the topic. The current formulation of the problem draws from each of these sources but follows the exposition of Clarke (1943) most closely.

2.1 Carson's Equations

Carson's formulas are

$$\mathbf{Z_{ii-g}} = r_i + j2\omega ln\left(\frac{2h_i}{gmr_i}\right) + 4\omega(P + jQ) \tag{1}$$

$$\mathbf{Z_{ij-g}} = j2\omega ln\left(\frac{D_{ij}}{d_{ij}}\right) + 4\omega(P + jQ) \tag{2}$$

where

$\mathbf{Z_{ii-g}}$ is the self-impedance of conductor i with ground return.

$\mathbf{Z_{ij-g}}$ is the mutual impedance between conductors i and j with common ground return.

gmr_i is the effective radius (or geometric mean radius) of conductor i in centimeters.

h_i is the height of conductor i in centimeters.

r_i is the internal resistance of conductor i.

d_{ij} the distance between conductors i and j in centimeters.

D_{ij} the distance between conductor i and the image of conductor j in centimeters.

ω is $2\pi f$, where f is the frequency in cycles per second.

Obviously, the self-impedance $\mathbf{Z_{ii-g}}$ and mutual impedance $\mathbf{Z_{ij-g}}$ can be decomposed into their real and imaginary components

$$\mathbf{Z_{ii-g}} = R_{ii-g} + jX_{ii-g} \tag{3}$$

$$\mathbf{Z_{ij-g}} = R_{ij-g} + jX_{ij-g} \tag{4}$$

Collecting terms in Equation 1 and Equation 2 and comparing to Equation 3 and Equation 4, it is apparent that

$$R_{ii-g} = r_i + 4\omega P \tag{5}$$

$$R_{ij-g} = 4\omega P \tag{6}$$

$$X_{ii-g} = 2\omega ln\left(\frac{2h_i}{gmr_j}\right) + 4\omega Q \tag{7}$$

$$X_{ij-g} = 2\omega ln\left(\frac{D_{ij}}{d_{ij}}\right) + 4\omega Q \tag{8}$$

$$\tag{9}$$

2.1.1 Approximation of P and Q in Carson's Equations

The P and Q terms in the preceding equations are defined by Carson as an infinite series expressed in terms of two parameters, call them k and θ. The form of P and Q are the same for Equation 1 and Equation 2. However, the value of k and θ differ. For self impedances

$$k = 4\pi h_i \sqrt{2\lambda f} \tag{10}$$
$$\theta = 0 \tag{11}$$

For mutual impedances

$$k = 2\pi D_{ij} \sqrt{2\lambda f} \tag{12}$$
$$\theta = \frac{cos^{-1}(h_i + h_j)}{D_{ij}} \tag{13}$$

where

λ is the earth conductivity in ab℧/cm³.

θ is the angle defined in Figure 1.

Figure 1 defines the line geometry associated with Equation 10 through Equation 13.

The first few terms of the expansion of P and Q follow:

$$\begin{aligned} P =& \frac{\pi}{8} - k\frac{\cos\theta}{3\sqrt{2}} + k^2\frac{\cos(2\theta)\left(0.6728 + \ln\left(\frac{2}{k}\right)\right)}{16} + k^2\frac{\theta\sin(2\theta)}{16} \\ &+ k^3\frac{\cos(3\theta)}{45\sqrt{2}} - k^4\frac{\pi\cos(4\theta)}{1536} \end{aligned} \tag{14}$$

$$\begin{aligned} Q =& -0.0386 + \frac{1}{2}ln\left(\frac{2}{k}\right) + k\frac{\cos\theta}{3\sqrt{2}} - \pi k^2\frac{\cos(2\theta)}{64} \\ &+ k^3\frac{\cos(3\theta)}{45\sqrt{2}} - k^4\frac{\sin(4\theta)}{384} - k^4\frac{\cos(4\theta)\left(1.0895 + \ln\left(\frac{2}{k}\right)\right)}{384} \end{aligned} \tag{15}$$

2.1.2 Accuracy of Approximations to P and Q

Clarke (1943) states that Equation 14 and Equation 15 exhibit less than one percent error for values of k up to one. Table 1 shows the wide applicability of these expressions for fundamental and harmonic analysis of power systems by examining values of k for a range of geometries, frequencies, and resistivities.

Figure 1: Transmission Line Geometry

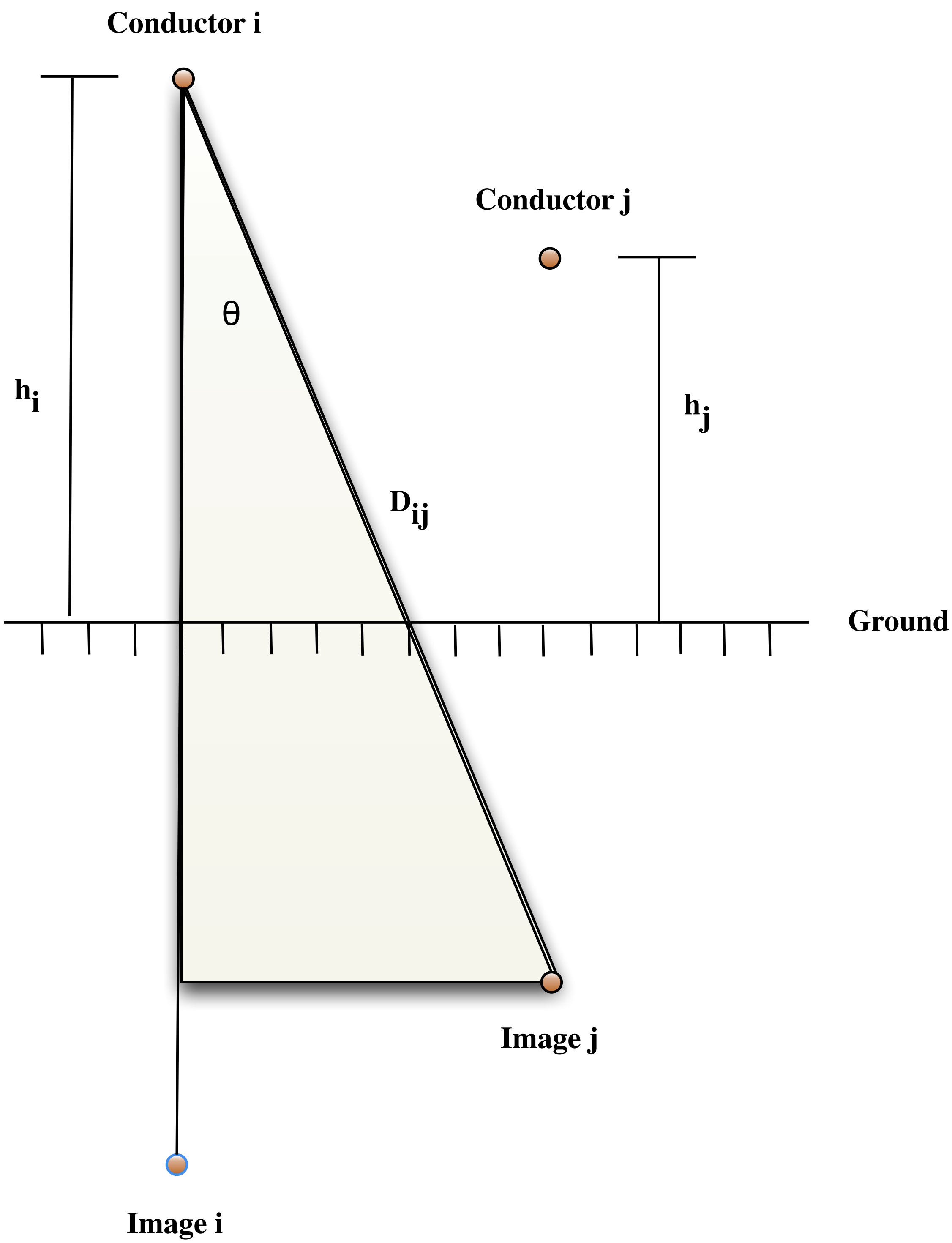

Table 1: Range of the Constant k

Distance	Frequency	Earth Resistivity	k
100 ft	60 Hz	10 Ω/m^3	0.4196
	660 Hz		1.3916
	1020 Hz		1.7300
	60 Hz	100 Ω/m^3	0.1327
	660 Hz		0.4401
	1020 Hz		0.5471
	60 Hz	1000 Ω/m^3	0.0419
	660 Hz		0.1391
	1020 Hz		0.1730

100 ft - Large double circuit transmission tower
10 Ω/m^3 - Resistivity of swampy ground
100 Ω/m^3 - Resistivity of average damp earth
1000 Ω/m^3 - Resistivity of dry earth

2.1.3 Use of First Order Approximations to P and Q

At 60 Hz, it is common practice to ignore the higher order terms of the expansion of P and Q, i.e. let

$$P = \frac{\pi}{8}$$
$$Q = -0.0386 + \frac{1}{2} ln\left(\frac{2}{k}\right)$$

This practice effectively decouples the series impedance from the conductor's height above ground. According to Wagner and Evans (1933), this omission tends to overstate the computed resistance and understate the computed reactance. At commercial frequencies and low earth resistivities (ρ=10), the first order approximations may introduce resistance errors in the neighborhood of 10 per cent. Under similar circumstances, self reactance errors rarely exceed one per cent. However, mutual reactance errors are more volatile. For ρ=10, f=60, and D_{ij}=200 feet, the low order approximation of Q understates the mutual reactance by much as 4 per cent. At higher harmonics, these tendencies are magnified.

2.2 Impedance of an N Conductor Transmission Line

The two conductor problem of Section 2.1 can be generalized to a group of n conductors with a common ground return. If currents i_1, i_2, ..., i_n are flowing through the conductors, the voltage drop along conductor i is

$$\mathbf{V_i} = i_1 Z_{i1-g} + \cdots + i_i Z_{ii-g} + \cdots + i_n Z_{in} \tag{16}$$

Similar equations can be constructed for all conductors in the group. Expressing the complete set of n voltage drop equations in matrix notation yields

$$\mathbf{V} = \mathbf{Z_{series}} \mathbf{I} \tag{17}$$

where

$\mathbf{V}$ is the voltage vector.

$\mathbf{I}$ is the current vector.

$\mathbf{Z_{series}}$ is the series impedance matrix.

The elements of the impedance matrix $\mathbf{Z_{series}}$ are computed using Carson's equations:

$$\mathbf{z_{ij}} = \begin{cases} R_{ii-g} + jX_{ii-g} & \text{if } i = j \\ R_{ij-g} + jX_{ij-g} & \text{if } i \neq j \end{cases} \tag{18}$$

where R_{ii-g}, R_{ij-g}, X_{ii-g}, and X_{ij-g} are defined by Equation 5 through Equation 8.

The series admittance of the n conductor configuration can be determined by inverting its impedance matrix, i.e.

$$\mathbf{Y_{series}} = \mathbf{Z_{series}^{-1}} \tag{19}$$

2.3 Series Impedance Computations

This discussion of overhead transmission line series impedance concludes with a brief dicussion of computing contant factors associated with the impedance matrix and reconciling units of measure while evaluating these constants.

2.3.1 Computation of k in the Series Approximation to P and Q

The parameter k appears in the series expansion which approximates the P and Q terms of Carson's equations (see Equation 10 and Equation 12 of Section 2.1.1 for details). It is of the form

$$k = 4\pi d\sqrt{2\lambda f} \tag{20}$$

where

λ is the earth conductivity in ab℧/cm³.

d is a distance in centimeters.

This can be rewritten in terms of readily available quantities (i.e. commonly published units) by substituting earth resistivity (Ω/m³) for conductivity and distance in conductor separation units for distance in centimeters as follows

$$k = 4\pi d(u_{CS} \rightarrow cm)\sqrt{\frac{2\lambda f(\lambda \rightarrow \rho)}{\rho}} \tag{21}$$

where

u_{CS} is conductor separation unit. In the US, conductor separation is usually measured in feet.

$u_{CS} \rightarrow$ cm is the number of centimeters per conductor separation unit.

$\lambda \rightarrow \rho$ is a constant converting ab℧/cm³ to Ω/m³.

Assuming that the frequency and resitivity are constant for any set of impedance computations the bulk of the expression

$$4\pi(u_{cs} \rightarrow cm)\sqrt{\frac{2\lambda f(\lambda \rightarrow \rho)}{\rho}} \tag{22}$$

is a constant which is computed once then stored for reuse.

2.3.2 Constants in the P and Q Terms of Carson's Equations

After P and $\mathcal{Q}$ are computed, the terms $4\omega P$ and $4\omega\mathcal{Q}$ in Equation 5 through Equation 8 of Section 2.1.1 produce impedances in units of abΩ/cm. If impedances are expressed in Ω/u_{LL}, these terms expand to

$$4\omega(u_{LL} \rightarrow cm)(ab\Omega \rightarrow \Omega)P \tag{23}$$

and

$$4\omega(u_{LL} \rightarrow cm)(ab\Omega \rightarrow \Omega)Q \tag{24}$$

where

u_{LL} is line length unit. In the US, line length is usually measured in miles.

$u_{CR} \rightarrow$ cm is the number of centimeters per line length unit.

abΩ → Ω is a constant converting abΩ to Ω, i.e. 1×10^{-9}.

Assuming that the frequency is constant, both P and $\mathcal{Q}$ are multiplied by the same factor

$$4 \cdot 2\pi f(u_{ll} \rightarrow cm)(ab\Omega \rightarrow \Omega) \tag{25}$$

The first terms of of the inductive reactance equations (Equation 7 and Equation 8 of Section 2.1) are also multiplied by half of this value, i.e.

$$2 \cdot 2\pi f(u_{ll} \rightarrow cm)(ab\Omega \rightarrow \Omega) \tag{26}$$

Once again, both of these constants are calculated once then stored.

2.3.3 Unit Conversions Associated With GMR Terms

When the logarithmic term in Equation 7 of Section 2.1 is computed, the conductor's GMR must be converted to conductor separation units, i.e.

$$ln\left(\frac{2h_i}{gmr_j}\right) \tag{27}$$

is actually evaluated as

$$ln\left(\frac{2h_i}{gmr_j(u_{CR} \rightarrow u_{CS})}\right) \tag{28}$$

where $u_{CR} \rightarrow u_{CS}$ converts conductor radius units to conductor separation units. Factoring out a constant in this expression yields

$$ln\left(\frac{ch_i}{gmr_j}\right) \tag{29}$$

where

$$c = \frac{2}{u_{CR} \rightarrow u_{CS}} \tag{30}$$

The factor c is also calculated once and stored.

3 Shunt Admittance

The capacitance of an overhead transmission line is primarily a function of conductor geometry and line geometry. All of the references cited in Section 2 with regard to series impedance also touch upon the subject of self and mutual capacitance. The current discussion most closely follows the work of Anderson (1987).

3.1 Linear Charge Density Along a Single Conductor

Assuming that a group of n conductors carrying linear charge densities q_1, q_2, ..., q_n are located above the ground plane, the voltage of conductor i to ground is

$$V_i = \frac{q_1 ln\left(\frac{D_{i1}}{d_{i1}}\right) + \cdots + q_i ln\left(\frac{D_{ii}}{d_i}\right) + \cdots + q_n ln\left(\frac{D_{in}}{d_{in}}\right)}{2\pi\epsilon} \tag{31}$$

where

q_i is the charge of conductor i in coulombs/meter.

d_i is the radius of conductor i.

D_{ii} is the distance between conductor i and its image (i.e. $2h_i$ in Figure 1).

d_{ij} is the distance between conductor i and conductor j.

D_{ij} is the distance between conductor i and the image of conductor j as illustrated in Figure 1.

ϵ is the permittivity of the medium.

Note: The distances associated with each logarithmic ratio (e.g. d_i and D_{ii} or d_{in} and D_{in}) of Equation 31 must be expressed in the same units.

3.2 Capacitance of N Conductors

Given a group of n conductors carrying linear charge densities q_1, q_2, ..., q_n that are located above the ground plane, equations of the same form as Equation 31 (Section 3.1) can be constructed for all conductors in the group. Expressing the complete set of n potential equations in matrix notation yields

$$\mathbf{V} = \mathbf{PQ} \tag{32}$$

where

$\mathbf{V}$ is the voltage vector.

$\mathbf{Q}$ is the charge vector.

$\mathbf{P}$ is the potential coefficient matrix.

The elements of the potential matrix (with units of $F^{-1}m$) are defined as follows:

$$p_{ij} = \begin{cases} \frac{ln\left(\frac{D_{ii}}{d_i}\right)}{2\pi\epsilon} & \text{if } i = j \\ \frac{ln\left(\frac{D_{ij}}{d_{ij}}\right)}{2\pi\epsilon} & \text{if } i \neq j \end{cases} \tag{33}$$

Recall that the permittivity of a medium is often expressed as

$$\epsilon = \epsilon_0 \epsilon_r \tag{34}$$

where

ϵ_0 is the permittivity of free space (i.e. $8.85541853 \times 10^{-12}$ F/m).

ϵ_r is the relative permittivity of the medium (e.g. 1 for air).

In matrix notation, the capacitance of the configuration is

$$\mathbf{Q} = \mathbf{C}\mathbf{V} \tag{35}$$

solving Equation 32 for the charge vector yields

$$\mathbf{Q} = \mathbf{P}^{-1}\mathbf{V} \tag{36}$$

By inspection it is apparent that

$$\mathbf{C} = \mathbf{P}^{-1} \tag{37}$$

The matrix **C** is sometimes known as the capacitance coefficients (or Maxwell's coefficients) of the line.

3.3 Shunt Admittance and Reactance Matrices

If the charge density along the transmission line is sinusoidal rather than linear, Equation 35 is a phasor equation. Multiplying Equation 35 by $j\omega$ yields

$$j\omega\mathbf{Q} = j\omega\mathbf{C}\mathbf{V} \tag{38}$$

Recalling that the current phasor associated with a sinusoidal variation in charge is expressed as

$$\mathbf{I} = j\omega\mathbf{Q} \tag{39}$$

It is apparent that

$$\mathbf{I} = j\omega\mathbf{C}\mathbf{V} \tag{40}$$

An alternate expression for the charging current is

$$\mathbf{I} = \mathbf{Y}_{\mathbf{shunt}}\mathbf{V} \tag{41}$$

Therefore, the charging admittance (which is pure susceptance) must be

$$\mathbf{Y}_{\mathbf{shunt}} = j\omega\mathbf{C} \tag{42}$$

The preceding discussion suggests a computational procedure for determining the capacitive parameters of a conductor configuration:

1. Compute the configuration's potential matrix $\mathbf{P}$ using Equation 33.
2. Compute its capacitance matrix $\mathbf{C}$ by inverting $\mathbf{P}$.
3. Multiply the capacitance matrix $\mathbf{C}$ by the scalar $j\omega$ to obtain the shunt admittance matrix $\mathbf{Y}_{\mathbf{shunt}}$.
4. Invert the the shunt admittance matrix $\mathbf{Y}_{\mathbf{shunt}}$ to determine the capacitive reactance $\mathbf{X}_{\mathbf{shunt}}$.

3.4 Shunt Admittance Computations

This discussion of overhead transmission line shunt admittance concludes with a brief dicussion of computing contant factors associated with the potential matrix and reconciling units of measure while evaluating these constants.

3.4.1 Potential Coefficient Unit Conversions

The constant associated with the computation of potential coefficients in Equation 33 depends only upon the medium in which the conductors reside. Assuming that the conductors are suspended in air ($\epsilon_r = 1$), the potential constant (in F⁻¹m) is

$$\frac{1}{2\pi\epsilon_0\epsilon_r} = \frac{1}{2\pi \cdot 8.8541853 \times 10^{-12} \cdot 1} = 1.79751087 \times 10^{10} \tag{43}$$

To compute potential coefficients in line length units rather than meters, an additional conversion factor $m \rightarrow u_{LL}$ is required, i.e. the multiplier in Equation 33 is actually

$$\frac{m \rightarrow u_{LL}}{2\pi\epsilon}$$

or

$$(m \rightarrow u_{LL})1.79751087 \times 10^{10} \tag{44}$$

which produces potential coefficients with units $F^{-1} \cdot u_{LL}$. This product is computed once and stored.

3.4.2 Self Potential Unit Conversions

The self potential in Equation 33 is

$$\frac{ln\left(\frac{D_{ii}}{d_i}\right)}{2\pi\epsilon}$$

When this term is computed, the numerator and denominator of the logarithmic factor must be in the same units. Assuming that the conductor's diameter (in u_{CD}) is readily available, the distance must be converted to conductor separation units and the diameter must be converted to a radius. Therefore, the computed logarithmic factor is

$$ln\left(\frac{D_{ii}}{\frac{d_i}{2}(u_{CD} \rightarrow u_{CS})}\right)$$

where $u_{CD} \rightarrow u_{CS}$ converts conductor diameter to conductor separation units.

Factoring out a constant in this expression

$$ln\left(c\frac{D_{ii}}{d_i}\right) \tag{45}$$

where

$$c = \frac{2}{u_{CR} \rightarrow u_{CS}} \tag{46}$$

The factor c is computed once and saved.

Note: In the context of the current discussion, the clear choice of unit for capacitive reactance is $\Omega \cdot u_{LL}$. However, the capacitive reactance found in American reference materials is often $M\Omega \cdot u_{LL}$ or more specifically $M\Omega \cdot$ mile. Hence, an additional factor may be required when converting capacitive reactance from computational units to commonly published units (ie. 10^{-6} for converting MΩ to Ω).

4 Units of Measure

A number of unit systems are involved in overhead transmission line impedance calculations. The current section is intended to make their distinctions clear. An engineer provides data for the calculations in what we will refer to as the "user" unit system (also referred to as "generally available units" or "commonly published units" in other sections of this document). User units may vary along the following lines:

- Conductor separation units, u_{CS}, are associated with conductor-to-conductor and conductor-to-image distances. In US applications, conductor separation is usually measured in feet.
- Conductor radius units, u_{CR}, are associated with effective radius (GMR) measurements. In American reference materials, conductor GMR is usually reported in feet.
- Conductor diameter units, u_{CD}, are associated with outside diameter measurements. In American reference materials, conductor diameter is usually reported in inches.
- Line length units, u_{LL}, are associated with the length of a line section. In US utility applications, line length is usually measured in miles.

The units in which equations are expressed in this document are called problem formulation units. The units in which impedance calculations are actually implemented are referred to as computation units.

Table 2 describes these unit systems in detail.

Table 2: Impedance Unit Conversions

	Units		
Quantity	Formulation	Computational	User
Frequency	Hz	Hz	Hz
Earth Resitivity	ab℧/cm³	Ω/m³	Ω/m³
k	abΩ/cm	$\Omega\,/\,u_{LL}$	n/a
Resistance	abΩ/cm	$\Omega\,/\,u_{LL}$	$\Omega\,/\,u_{LL}$
Inductive Reactance	abΩ/cm	$\Omega\,/\,u_{LL}$	$\Omega\,/\,u_{LL}$
Potential Coefficients	$\mathrm{F}^{-1}\cdot\mathrm{m}$	$\mathrm{F}^{-1}\cdot u_{LL}$	n/a
Maxwell's Coefficients	F/m	$\mathrm{F}\,/\,u_{LL}$	n/a
Capacitive Susceptance	℧/m	$\mho\,/\,u_{LL}$	n/a
Capacitive Reactance	$\Omega\cdot\mathrm{m}$	$\Omega\cdot u_{LL}$	$(\mathrm{M})\Omega\cdot u_{LL}$
Conductor Diameter	m	u_{CS}	u_{CD}
Conductor GMR	cm	u_{CS}	u_{CR}
Conductor Separation	cm	u_{CS}	u_{CS}

u_{CD} - conductor diameter unit
u_{CR} - conductor radius unit
u_{CS} - conductor separation unit
u_{LL} - line length unit

References

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